

Spin chains of Haldane-Shastry type and a generalized central limit theorem

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We show that the density of energy levels of a wide class of finite-dimensional quantum systems tends to a Gaussian distribution as the number of degrees of freedom increases. Our result is based on a variant of the central limit theorem which is especially suited to models whose partition function is explicitly known. In particular, we provide a theoretical explanation of the fact that the level density of several spin chains of Haldane-Shastry type is asymptotically Gaussian when the number of sites tends to infinity.

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A widely accepted tool for detecting chaos in a quantum system is the behavior of the distribution of the spacings between consecutive levels [1]. In fact, a long-standing conjecture of Berry and Tabor [2] posits that the spacings distribution of a “generic” integrable system should be Poissonian, while in quantum chaotic systems like polygonal billiards this distribution is usually given by Wigner’s surmise [3], characteristic of the Gaussian ensembles in random matrix theory. An essential prerequisite for computing the spacings distribution is the knowledge of the continuous part of the density of energy levels, whose distribution function is used to rescale the original spectrum so that the transformed (“unfolded”) energies have a uniform local density. The identification of the continuous part of the level density is in general a delicate numerical procedure, although in a few cases it can be performed from first principles. For instance, it is known that the continuous part of the level density of a large random matrix in the Gaussian orthogonal ensemble is approximately given by Wigner’s circular law [3], whereas in many quantum billiards the cumulative level density can be computed using Weyl’s law [1].

An important class of quantum systems which has been used to test Berry and Tabor’s conjecture are spin chains of Haldane-Shastry (HS) type, which include the original HS chain [4,5] and its rational version studied by Polychronakos and Frahm [6,7] (both based on the A_{N-1} root system in the Olshanetsky and Perelomov scheme [8]), as well as their generalizations associated with the BC_N and D_N root systems [9–12] and their supersymmetric variants [13–15]. These chains, which are the prime examples of integrable spin chains with long-range interactions, have close connections with several topics of current interest such as strongly correlated systems [16], generalized exclusion statistics [17,18], and the anti de Sitter space-conformal field theory (AdS-CFT) correspondence [19]. Recent numerical computations suggest that a common feature of these chains [10,12,15,20–24] is the fact that the level density (normalized to 1) becomes Gaussian when the number of sites N

tends to infinity. For many spin chains of HS type, it has been shown that the Gaussian character of the level density implies that the spacings distribution obeys neither Poisson’s nor Wigner’s law but is rather given by a simple “square root of a logarithm” formula [12,15,21,22,24]. It would thus be desirable to provide a theoretical explanation of the fact that the level density of these models is approximately Gaussian for large N , which so far has only been checked numerically.

In this Rapid Communication, we develop a generalization of the standard central limit theorem to show that the level density of a wide class of finite-dimensional quantum systems must be asymptotically Gaussian. This class includes in particular a supersymmetric version of the original (trigonometric) Haldane-Shastry spin chain, as well as the BC_N version of the spin-1/2 Polychronakos-Frahm (rational) chain. Although our result does not apply to all spin chains of HS type, we believe that the ideas on which it relies should play a key role when addressing the general case.

To explain the gist of our approach, let us begin by considering an extremely simple toy model for which the asymptotically Gaussian character of the spectrum can be easily established. Indeed, let \mathcal{H} be a finite-dimensional Hilbert space and let us denote by $\mathcal{H}^{\otimes N}$ its N th tensor product, which can be identified with the N -particle space. Our toy Hamiltonian will be given by

$$H = I_1 + \cdots + I_N, \quad (1)$$

where each operator $I_k = 1_{\mathcal{H}} \otimes \cdots \otimes \overset{k}{I} \otimes \cdots \otimes 1_{\mathcal{H}}$ is a constant of motion acting on the Hilbert space of the k th particle as some fixed operator I . Physically, this system can be thought of as consisting of N identical noninteracting particles. Had we allowed the one-particle Hilbert space to be infinite dimensional, this would precisely be the structure underlying the N -dimensional harmonic oscillator.

It is not difficult to see that in the limit $N \rightarrow \infty$ the level density of the Hamiltonian (1) becomes Gaussian, with mean $\mu = N\mu_I$ and variance $\sigma^2 = N\sigma_I^2$ (μ_I and σ_I^2 , respectively, denoting the mean and variance of the spectrum of I). Indeed, the eigenvalues of H are given by

$$\mathcal{E} = \mathcal{E}_1 + \cdots + \mathcal{E}_N, \quad (2)$$

where each \mathcal{E}_k is an eigenvalue of the operator I . Since each

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of these eigenvalues can be freely chosen, the parameters \mathcal{E}_k in the above formula can be regarded as independent random variables with the same distribution. Thus, the fact that the random variable \mathcal{E} asymptotically follows the Gaussian law as $N \rightarrow \infty$ is an immediate consequence of the central limit theorem.

It is natural to wonder whether the simple argument above can be extended to a wider class of models. A cursory inspection reveals, however, that several assumptions must be substantially relaxed in order to cover any physically interesting situation. In particular, in Eq. (2) it is crucial to allow for sums of independent random variables which are *not* identically distributed. It is also clear that it would be preferable to express the conditions on the spectrum of our model in terms of its partition function rather than its eigenvalues. Indeed, for chains of HS type, the partition function can be computed in closed form through Polychronakos's "freezing trick" [6,25], while the spectrum is considerably more difficult to handle. Thus, in order to further develop the former approach, we will need to prove a modification of the classical central limit theorem incorporating the above requirements.

Let us introduce some notation before going on. We shall denote by $Z(q) = \sum_{i=1}^n d_i q^{\mathcal{E}_i}$ the partition function of a finite-dimensional Hamiltonian H with energy levels \mathcal{E}_i and degeneracies d_i , $1 \leq i \leq n$, where $q = e^{-1/(k_B T)}$ and k_B is Boltzmann's constant. The value $Z(1) = \sum_{i=1}^n d_i$ is thus the dimension of the Hilbert space of the system. Its thermodynamic energy at temperature T is given by $E(q) = q \partial / \partial q \ln Z(q)$, in terms of which the mean and variance of the spectrum of H can be expressed as

$$\mu = \langle H \rangle \equiv \frac{1}{Z(1)} \sum_{i=1}^n d_i \mathcal{E}_i = E(1),$$

$$\sigma^2 = \langle (H - \mu)^2 \rangle \equiv \frac{1}{Z(1)} \sum_{i=1}^n d_i (\mathcal{E}_i - \mu)^2 = E'(1).$$

We will henceforth restrict our attention to systems whose partition function can be written as a product

$$Z(q) = \prod_{k=1}^N Z_k(q; N), \quad (3)$$

where we have assumed that the factors Z_k may depend on the integer parameter N in view of forthcoming applications. Roughly speaking, the system is equivalent to an effective model of N noninteracting but not necessarily identical subsystems. Two powerful tools in the study of the distribution of the energy levels of Eq. (3) are the characteristic function

$$\varphi(t) \equiv \langle e^{itH} \rangle = Z(e^{it}) / Z(1), \quad (4)$$

which is essentially its Fourier transform, and the normalized characteristic function

$$\bar{\varphi}(t) \equiv \langle e^{it(H-\mu)/\sigma} \rangle = e^{-it\mu/\sigma} \varphi(t/\sigma). \quad (5)$$

The key property of the latter function is that under very mild technical assumptions [26], in the limit $N \rightarrow \infty$ the level density of the system asymptotically follows the Gaussian

law with mean μ and standard deviation σ if and only if

$$\lim_{N \rightarrow \infty} \bar{\varphi}(t) = e^{-t^2/2}. \quad (6)$$

We shall next provide some simple conditions on the factors Z_k ensuring that Eq. (6) holds. In order to state them in a concise way, we will denote by $\bar{\varphi}_k(t; N)$ the normalized characteristic function of the k th subsystem and introduce the notation

$$\mathcal{M}_k(\tau; N) \equiv \sup_{|s| < \tau} |(\partial^3 / \partial s^3) \ln \bar{\varphi}_k(s; N)|.$$

It should be noticed that the average and the standard deviation of the energy of the whole system are related to the analogous quantities $\mu_k(N)$ and $\sigma_k(N)$ of its subsystems by

$$\mu = \sum_{k=1}^N \mu_k(N), \quad \sigma^2 = \sum_{k=1}^N \sigma_k^2(N). \quad (7)$$

For the sake of conciseness, from now on we shall often omit the dependence on N of Z_k , $\bar{\varphi}_k$, μ_k , and σ_k . We shall make the following assumptions on the factors:

(i) The standard deviation of the full partition function and of its k th factor satisfy

$$\sigma_k / \sigma \leq C_1 N^{-1/2}, \quad 1 \leq k \leq N, \quad (8)$$

where the constant C_1 does not depend on N .

(ii) There exist some positive constants $\epsilon_1, \epsilon_2, C_2$, independent of N , such that

$$\mathcal{M}_k(\epsilon_1) \leq C_2 N^{1/2 - \epsilon_2}, \quad 1 \leq k \leq N. \quad (9)$$

Let us briefly discuss the meaning of these assumptions. Roughly speaking, the first condition holds whenever the standard deviation of the whole system does not effectively depend only on a few subsystems. More precisely, let $M(N) \equiv \max\{\sigma_k^2 : k=1, \dots, N\}$, and suppose that there exists a constant $\alpha > 0$ (independent of N) and a function $m(N) > 0$ such that $\sigma_k^2 \geq m(N)$, for at least αN values of k . Then, a sufficient condition ensuring the validity of (i) is that there exists a constant A such that

$$M(N) \leq Am(N) \quad (10)$$

as $N \rightarrow \infty$. Indeed, by definition of $m(N)$ and Eq. (7), we have that $\sigma^2 \geq \alpha N \cdot m(N)$, which in turn implies that $\sigma_k / \sigma \leq (\frac{M(N)}{\alpha N m(N)})^{1/2} \leq C_1 N^{-1/2}$ for $1 \leq k \leq N$, with $C_1 = (A/\alpha)^{1/2}$. On the other hand, condition (ii) is a local bound on the third central moment of the thermodynamic energy of the k th subsystem at imaginary temperature. Indeed, it can be shown that

$$\begin{aligned} \frac{1}{i^3} \frac{\partial}{\partial s^3} \ln \bar{\varphi}_k(s) &= \frac{1}{i^3} \frac{\partial}{\partial s^3} \ln Z_k(e^{is/\sigma_k}) = \left(\frac{q}{\sigma_k} \frac{\partial}{\partial q} \right)^3 \ln Z_k(q) \Big|_{q=e^{is/\sigma_k}} \\ &= \frac{1}{\sigma_k^3} \overline{[\mathcal{E}_k - E_k(q)]^3} \Big|_{q=e^{is/\sigma_k}}, \end{aligned}$$

where the overbar denotes the thermal average.

It should also be noted that our second condition is reminiscent of Lyapunov's classical condition [26], according to which Eq. (6) holds if there is a positive δ such that

$$\lim_{N \rightarrow \infty} \frac{1}{\sigma^{2+\delta}} \sum_{k=1}^N \langle |\mathcal{E}_k - \mu_k|^{2+\delta} \rangle = 0. \quad (11)$$

Note, however, that due to the absolute value the latter condition cannot be expressed in terms of the partition functions Z_k (this is also true for the more general condition due to Lindeberg [26]). Thus, Eq. (11) may be impractical when Z_k is known but there is not an explicit formula for its spectrum [27].

In order to see that conditions (i) and (ii) above imply that the level density asymptotically follows the Gaussian law, let us compute the limit of $\ln \bar{\varphi}(t)$ as $N \rightarrow \infty$. To begin with, one should observe that the fact that $\bar{\varphi}_k$ is normalized to zero mean and unit variance ensures that the second-order Taylor expansion of $\ln \bar{\varphi}_k$ around 0 is $\ln \bar{\varphi}_k(\tau) = -\tau^2/2 + R_k(\tau)$, where the remainder is bounded by

$$|R_k(\tau)| \leq (|\tau|^3/6) \mathcal{M}_k(|\tau|). \quad (12)$$

Since $\varphi(t) = \prod_{k=1}^N \varphi_k(t)$ by Eq. (4), it immediately follows from Eqs. (5) and (7) that

$$\ln \bar{\varphi}(t) = \sum_{k=1}^N \ln \bar{\varphi}_k\left(\frac{\sigma_k t}{\sigma}\right) = -\frac{t^2}{2} + \sum_{k=1}^N R_k\left(\frac{\sigma_k t}{\sigma}\right),$$

where by virtue of Eq. (12) the error can be estimated as

$$\left| \ln \bar{\varphi}(t) + \frac{t^2}{2} \right| \leq \frac{1}{6} \sum_{k=1}^N \left(\frac{\sigma_k |t|}{\sigma} \right)^3 \mathcal{M}_k\left(\frac{\sigma_k |t|}{\sigma}\right). \quad (13)$$

Let us assume that $|t| < t_0$, where t_0 is a fixed but otherwise arbitrary constant, and take N greater than $(\frac{C_1 t_0}{\epsilon_1})^2$. In this case, by Eq. (8) we have $\sigma_k |t| / \sigma \leq C_1 t_0 N^{-1/2} \leq \epsilon_1$, so that Eq. (13) can be controlled as $|\ln \bar{\varphi}(t) + t^2/2| \leq \frac{1}{6} (C_1 t_0)^3 C_2 N^{-\epsilon_2}$, on account of Eq. (9). It then follows that $\ln \bar{\varphi}(t)$ converges pointwise to $-\frac{1}{2}t^2$ for all real t , the convergence being uniform on compact sets. Hence Eq. (6) holds, and thus the level density becomes asymptotically Gaussian by the properties of characteristic functions, as we wanted to show.

A particularly simple class of partition functions of the form (3) is obtained by requiring that each factor Z_k corresponds to a two-level system. In this case, setting the ground-state energy to zero without loss of generality, each factor can be written as

$$Z_k(q; N) = 1 + q^{\mathcal{E}(k, N)}, \quad (14)$$

and the mean and the standard deviation of its energy are readily computed as $\mu_k = \sigma_k = \frac{1}{2} \mathcal{E}(k, N)$. Its reduced characteristic function $\bar{\varphi}_k(\tau) = e^{-i\mu_k \tau} \sigma_k \frac{Z_k(e^{i\tau \sigma_k})}{Z_k(1)} = \frac{1}{2} e^{-i\tau} (1 + e^{2i\tau}) = \cos \tau$ is independent of both k and N , so that condition (ii) is automatically satisfied. It can be shown that condition (i) is satisfied as well, e.g., whenever $\mathcal{E}(k, N)$ depends polynomially on k and N . More precisely, if $r = \deg \mathcal{E}$ then $\sigma^2 \sim N^{2r+1}$, while σ_k^2 is at most $O(N^{2r})$.

We shall next discuss how the previous developments can be directly applied to two specific spin chains of HS type, which are of considerable interest in themselves. The first one is the Polychronakos-Frahm chain of BC_N type [11], whose Hamiltonian is given by

$$H = \sum_{\substack{j, k=1 \\ j \neq k}}^N \left[\frac{1 + \epsilon S_{jk}}{(\xi_j - \xi_k)^2} + \frac{1 + \epsilon S_j S_k S_{jk}}{(\xi_j + \xi_k)^2} \right] + \beta \sum_{k=1}^N \frac{1 - \epsilon' S_k}{\xi_k^2},$$

where $\epsilon^2 = \epsilon'^2 = 1$, $\beta > 0$, S_{jk} is the operator that permutes the j th and k th spins, and S_k is the operator flipping the k th spin. Moreover, the chain site ξ_k is expressed in terms of the k th zero y_k of the generalized Laguerre polynomial $L_N^{\beta-1}$ as $\xi_k = \sqrt{2} y_k$. It has recently been shown [21] that for spin 1/2 the partition function of this model is given by

$$Z(q) = q^{\delta_1 \epsilon^{N(N-1)/2}} \prod_{k=1}^N (1 + q^k), \quad (15)$$

where $\delta_1 \epsilon$ is the Kronecker delta. Other than the inessential factor $q^{\delta_1 \epsilon^{N(N-1)/2}}$, which can be removed by shifting the ground-state energy, this partition function is precisely of the form (14) with $\mathcal{E}(k, N) = k$ independent of N . It follows immediately from our previous discussion that when $N \rightarrow \infty$ the spectrum of H is normally distributed, with mean and variance given by

$$\mu = \sum_{k=1}^N \frac{k}{2} = \frac{N}{4} (N+1), \quad (16)$$

$$\sigma^2 = \sum_{k=1}^N \frac{k^2}{4} = \frac{N}{12} \left(N + \frac{1}{2} \right) (N+1). \quad (17)$$

This fact had been numerically verified in Ref. [21].

The above result has an interesting interpretation in classical partition theory. Indeed, by Eq. (15) the energies of the ferromagnetic chain ($\epsilon = -1$) are the integers in the range $0, 1, \dots, N(N+1)/2$, the degeneracy of an energy k being the number $Q_N(k)$ of partitions of the integer k into distinct parts not larger than N [with $Q_N(0) \equiv 1$]. We have thus established the large N asymptotic formula $Q_N(k) \sim \frac{2^N}{\sqrt{2\pi\sigma}} e^{-(k-\mu)^2/(2\sigma^2)}$, with μ, σ given in Eqs. (16) and (17).

The second spin chain we shall consider is the supersymmetric version of the celebrated Haldane-Shastry chain, with Hamiltonian given by [13]

$$H = \frac{1}{2} \sum_{\substack{j, k=1 \\ j < k}}^N \frac{1 + \epsilon P_{jk}}{\sin^2(\vartheta_j - \vartheta_k)}, \quad \vartheta_k \equiv \frac{k\pi}{N}, \quad (18)$$

where $\epsilon = \pm 1$. The supersymmetric spin permutation operator P_{jk} acts on an element $|s_1, \dots, s_N\rangle$ of the spin basis as $P_{jk}|s_1, \dots, s_j, \dots, s_k, \dots\rangle = \epsilon_{jk}(\mathbf{s}) |s_1, \dots, s_k, \dots, s_j, \dots\rangle$, where $\epsilon_{jk}(\mathbf{s})$ is -1 when either both s_j and s_k are fermionic spins, or s_j and s_k are spins of different type with an odd number of fermionic spins between them, and 1 otherwise. In the $su(1|1)$ case (i.e., when there is only one bosonic and one fermionic internal degree of freedom), the partition function of the chain (18) can be written as [28]

$$Z(q) = 2 \prod_{k=1}^{N-1} (1 + q^{k(N-k)}),$$

for both $\epsilon = \pm 1$. Apart from the irrelevant factor of 2 (which does not affect the level density, since it is normalized to 1), this partition function is a product of $N-1$ factors of the form (14) with a polynomial function $\mathcal{E}(k, N) = k(N-k)$. Hence, the above argument rigorously establishes that when $N \rightarrow \infty$ the level density of the Hamiltonian (18) becomes Gaussian, with parameters

$$\mu = \frac{1}{2} \sum_{k=1}^{N-1} k(N-k) = \frac{N}{12} (N^2 - 1),$$

$$\sigma^2 = \frac{1}{4} \sum_{k=1}^{N-1} k^2(N-k)^2 = \frac{N}{120} (N^4 - 1),$$

in whole agreement with the numerical computations in Ref. [23].

To conclude, let us summarize our results and offer some perspectives. We have provided conditions on the partition function of a finite-dimensional quantum system depending on a positive integer N , ensuring that its level density is asymptotically Gaussian as $N \rightarrow \infty$. Our conditions, which are related to Lyapunov's generalization of the classical central limit theorem, are directly formulated in terms of the partition function and do not require the explicit knowledge of

the spectrum. We have applied our result to rigorously show that the level density of two well-known spin chains of HS type becomes Gaussian as the number of sites tends to infinity. The first chain discussed is associated with the BC_N root system and presents rational interactions, while the second one is a supersymmetric version of the original (A_{N-1} type) Haldane-Shastry spin chain.

Our result does not apply to all spin chains of HS type, since the partition function of these models in general does not factorize as in Eq. (3). What seems to be true in this case, however, is that there is an approximate factorization

$$Z(q) = [1 + \epsilon(N, q)] \prod_{k=1}^N Z_k(q), \quad (19)$$

where the error term $\epsilon(N, q)$ and its q derivatives tend to 0 in a controlled way as the number of sites N tends to infinity. Although we shall not further elaborate on this point here, it is clear that the overall factor $1 + \epsilon(N, q)$ can be taken into account in our approach by imposing appropriate technical conditions analogous to Eqs. (8) and (9). We shall provide a more detailed discussion of this issue in a forthcoming paper.

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